Radiation pattern of a focused transducer: A numerically convergent solution

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The radiation pattern of a focused transducer is reexamined. The radiation field is divided into an illuminated zone and a shadow zone. A numerically convergent solution of the pressure distribution in terms summations of Bessel functions is provided. This solution is
[illegible]

When the size of the transducer is larger than the acoustic

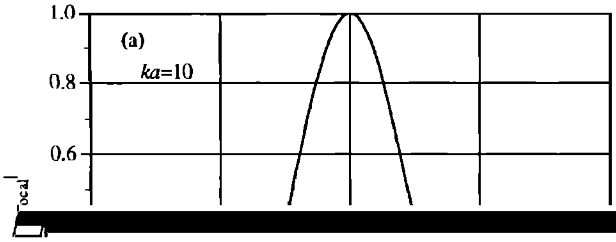
$$\phi=\frac{u_0 r_0^2}{r}\exp(-ikr)\int_{\theta_0=0}^{\alpha}\exp\left[-i\frac{kr_0^2}{r}\left(1-\frac{r}{r_0}\cos\theta\right)\right.$$

$$\left.\times(1-\cos\theta_0)\right]\sin\theta_0 J_0(kr_0\sin\theta\sin\theta_0)\,d\theta_0 \quad (8)$$

The two-parameter problem similar to that given in Eq. (16) has been discussed in detail by Lommel.[8] The following two equations are given (Eq. 32 in Ref. 8):

x ↑ [illegible] $\left(iZ^2\right)$ [illegible] $\left(Y \quad Z^2\right)$ [illegible]

h [illegible]



B. On the hemispherical surface $r=r_0$

On the hemispherical surface passing through the focal point $(r=r_0)$, we find $Y=4G_p \sin^2 \theta/2$ and $Y/Z=(a/r_0)\tan \theta/2$. Therefore,

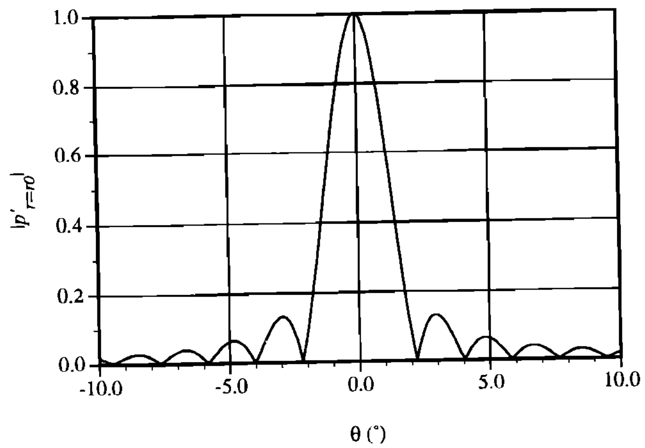


FIG. 4. Normalized pressure distribution on the surface $r=r_0$ for a focused transducer with $\sin\alpha=0.1$ and $ka=100$ $(G_p=5)$.

Therefore the normalized pressure distribution is

$$p'_{\text{axial}}=\exp[-ik(z-r_0)]\,\frac{r_0}{z}\,\frac{1-\exp[-iG_p(r_0/z-1)]}{iG_p(r_0/z-1)},\tag{50}$$

and its amplitude is

$$|p'_{\text{axial}}|=\frac{r_0}{z}\left|\frac{\sin[(G/2)(r_0/z-1)]}{(G/2)(r_0/z-1)}\right|.\tag{51}$$

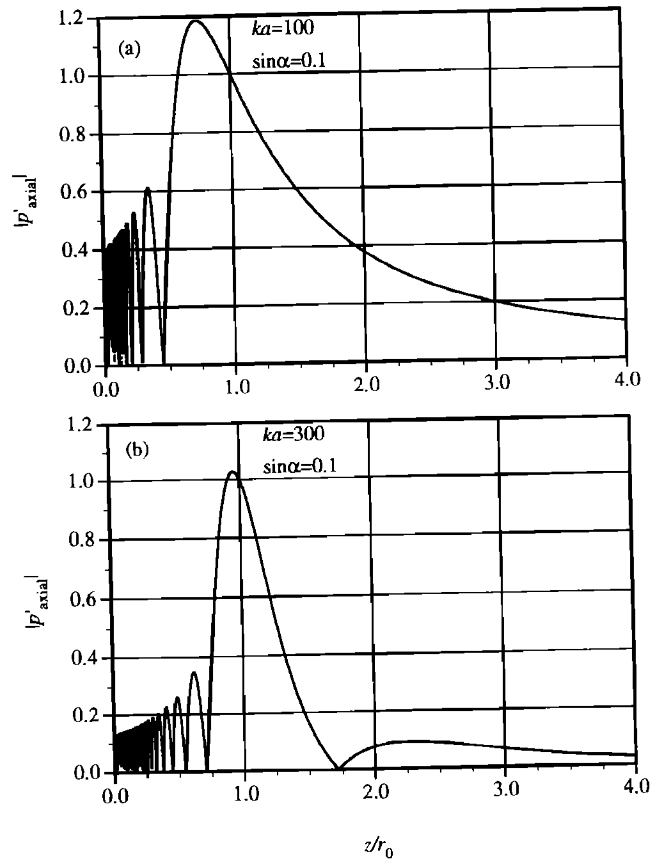


FIG. 5. Normalized axial pressure distribution of a focused transducer [redacted] with (a) $\sin\alpha=0.1$, $ka=100$ $(G_p=5)$; (b) $\sin\alpha=0.1$, $ka=300$ [redacted]

From Eq. (51) the pressure maximum on the axis occurs [redacted]

at

$$\frac{z_{\max}}{r_0}=1-\frac{12}{G_p^2}+O\left(\frac{1}{G_p^4}\right).\tag{52}$$

This agrees with the result obtained by Lucas and Muir.[6]

$$\phi_{\text{axial}}=\frac{u_0}{ik}\exp(-ikz)\,\frac{r_0}{z}\left[1-\exp\left(-ikz\left\{\left[1+\frac{4r_0}{z}\right.\right.\right.\right.$$

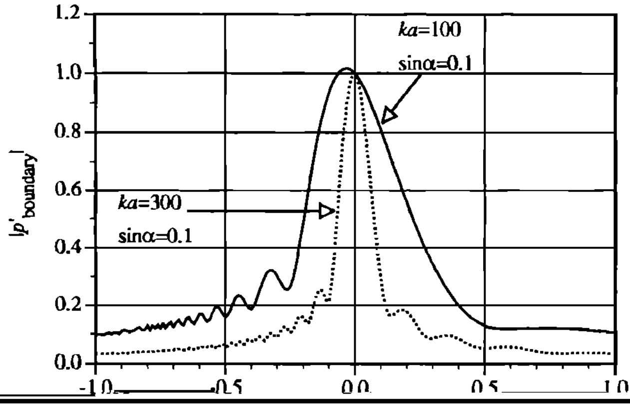


FIG. 6. Normalized pressure distribution on the boundary line for a focused transducer with $\sin\alpha=0.1$, $ka=100$ ($G_p=5$) and $\sin\alpha=0.1$, $ka=300$ ($G_p=15$).

From Eq. (58), the pressure node beyond the focal point is actually located at

$$\frac{z_n'}{r_0}=\frac{1-(2n\pi/ka)^2}{1-2n\pi/G_p},\quad n=1,2,\ldots<\frac{G_p}{2\pi}. \tag{59}$$

Equation (59) also gives all the zeroes before the focal

ary line for $\sin\alpha=0.1$ and $ka=100$ and 300. From Eq. (62) it can be proven that there are no pressure nodes along the boundary line for any values of ka.

At the edge of the transducer, where $r\approx a$ and $\theta\approx\pi/2$, the pressure amplitude is

$$|p_{\mathrm{edge}}|\approx(p_0/2)\left[1+J_0^2(ka)-2\cos(ka)J_0(ka)\right]^{1/2}. \tag{64}$$

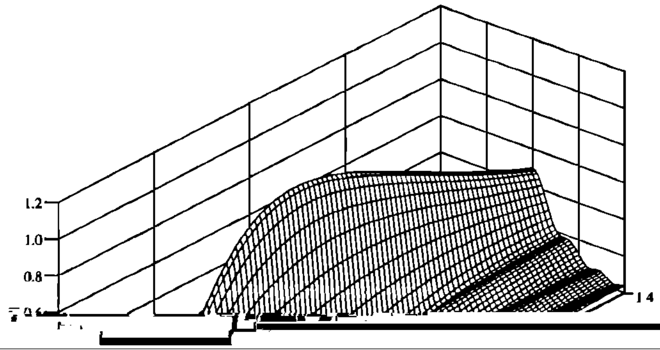
This expression indicates that the pressure amplitude at the edge of the transducer is roughly half of the power equivalent average pressure radiated from the transducer. Similar phenomena has been observed for a flat piston

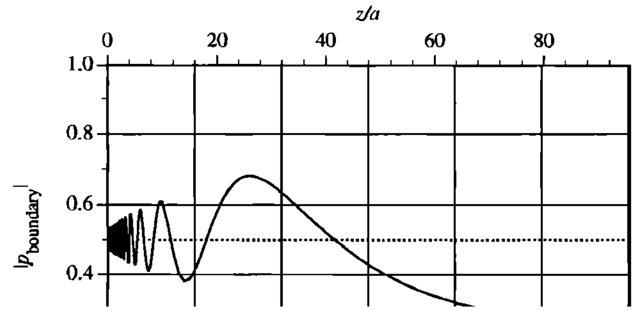
E. The physical meaning of p_0

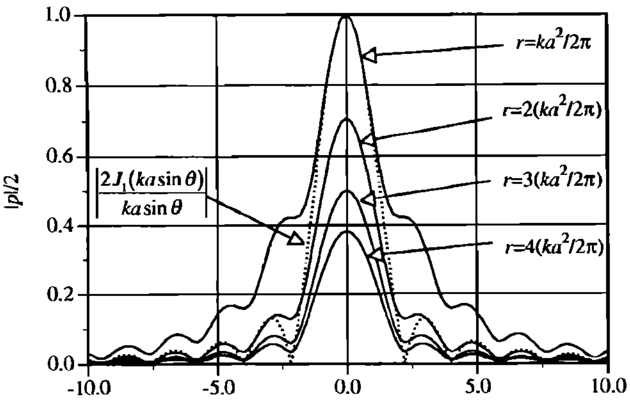
Since the pressure near the center of the transducer can be twice the value of p_0, and the pressure at the edge of the transducer is about half the value of p_0, the physical meaning of p_0 is desired. The total transmitted acoustic power can be computed as

$$P=\frac{1}{2\rho_0 c}\iint|p|^2\,dS, \tag{65}$$

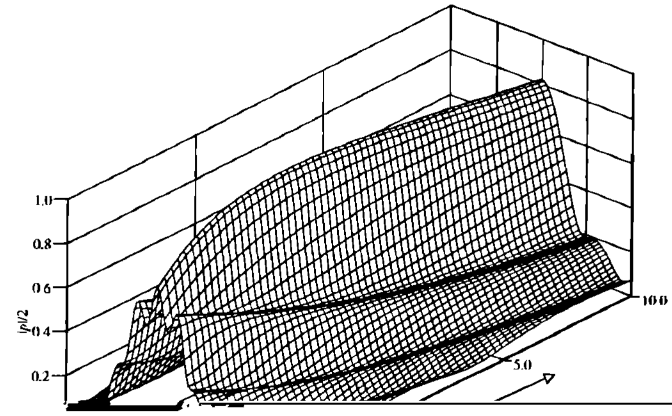
where the integral is chosen to be over the focal plane,


1.2
1.0
0.8
1.4


z/a
0
20
40
60
80
1.0
0.8
0.6
0.4
|p_boundary|



1.0
0.8
0.6
0.4
0.2
0.0
-10.0
-5.0
0.0
5.0
10.0
|p|/2
r=ka^2/2π
r=2(ka^2/2π)
r=3(ka^2/2π)
r=4(ka^2/2π)
|2J_1(ka sin θ) / ka sin θ|



1.0
0.8
0.6
0.4
0.2
|p|/2
10.0
5.0

$$-\sum_{n=0}^{n=\infty}(-1)^n\left[J_{2n+2}(Y)\cos E_n\left(\frac{Z^2-Y^2}{2Y}\right)\right.$$

[redacted]

where

$$\cos E_n(x)=\cos(x)-\sum_{q=0}^{q=n}(-1)^q\frac{1}{(2q)!}x^{2q},$$

$$\sin E_n(x)=\sin(x)-\sum_{q=0}^{q=n}(-1)^q\frac{1}{(2q+1)!}x^{2q+1}, \tag{78}$$

are the error functions of $\cos(x)$ and $\sin(x)$ when their Taylor expansions are truncated to order n, and an exten-

[redacted]

specified. For $|Y|<4\pi$ and $N=7$, we find $\varepsilon<0.01$.

The above discussion is only a rough estimate of terms needed. The actual number of terms needed is much smaller to obtain the degree of precision specified. For ex-

[redacted]

plitude, $N=0$ is needed to obtain $\varepsilon<0.01$.

Due to the nature of the series solution, recursive method should be used to calculate the values of the Bessel functions, since values of $J_n(Z)$ for roughly $n=0,\ldots,2N+2$ are needed when any of the three solutions are used. Also, since most of the computation time is spent on calculating the values of Bessel functions, it is advised that polar co-

ordinate systems be used. Then for a particular transducer (ka=const), the argument for the Bessel functions is a

[redacted]

[redacted]

Transducer type	Focused transducer		Flat disk transducer	
Parameters	$Y=\frac{ka^2}{r}\left(1-\frac{r}{r_0}\cos\theta\right)$, $Z=ka\sin\theta$.	Eq. (15)	$Y=ka^2/r$, $Z=ka\sin\theta$.	Eq. (67)
Pressure distribution	$p(r,\theta)=\frac{ip_0\exp(-ikr)}{1-r\cos\theta/r_0}I(Y,Z)$	Eq. (17)	$p(r,\theta)=ip_0\exp(-ikr)I(Y,Z)$.	Eq. (68)
Zones	$\left\lvert\frac{Y}{Z}\right\rvert=\begin{cases}<1, & \text{shadow } (\alpha<\theta_1<\pi-\alpha),\\ =1, & \text{boundary } (\theta_1=\alpha \text{ or } \theta_1=\pi-\alpha),\\ >1, & \text{illuminated } (\theta_1<\alpha \text{ or } \theta_1<\pi-\alpha).\end{cases}$	Eq. (25)	$\frac{Y}{Z}=\begin{cases}<1, & \text{shadow } (x>a),\\ =1, & \text{boundary } (x=a),\\ >1, & \text{illiminated } (x<a).\end{cases}$	Eq. (69)
General solution	$I(Y,Z)=Y\int_{u=0}^{1}\exp\left(-i\frac{Y}{2}u^2\right)uJ_0(Zu)\,du$			Eq. (16)
Shadow zone	$I(Y,Z)=\exp\left(-i\frac{Y}{2}\right)[u_1(Y,Z)+iu_2(Y,Z)]$			Eq. (21)
	$u_1(Y,Z)$, $u_2(Y,Z)$			Eq. (19)
Boundary	$I(\pm Z,Z)=\exp\left(\frac{\pm iZ}{2}\right)\frac{1-\exp(\mp iZ)j_0(Z)}{\pm 2i}$			Eq. (27)
Illuminated	$I(Y,Z)=-i\exp\left(\frac{iZ^2}{\ldots}\right)\left\{1-\exp\left[-i\left(\frac{Y}{\ldots}+\frac{Z^2}{\ldots}\right)\right][v_0(Y,Z)+iv_1(Y,Z)]\right\}$			Eq. (31)

[redacted]

$\sum^{\infty} \partial^q u_s(Y,Z_0)$ ... $(J_0(Y)+\cos(Y)$

Using the partial derivatives of $u_s(Y,Z)$ given by Gray and Mathews,[8] we have

$$u_s(Y,Z_0+\Delta Z)=\sum_{q=0}^{\infty}(-1)^q\times\frac{(\Delta Z)^q(2Z_0+\Delta Z)^q}{q!(2Y)^q}\times u_{s+q}(Y,Z_0). \tag{A3}$$

Let $Z=Z_0+\Delta Z$, and $Z_0=|Y|$, then $\Delta Z=Z-|Y|$, and Eq. (A3) becomes

$$u_s(Y,Z)=\sum_{q=0}^{\infty}(-1)^q\frac{1}{q!}\left(\frac{Z^2-Y^2}{2Y}\right)^q u_{s+q}(Y,|Y|). \tag{A4}$$

The functions $u_s(Y,|Y|)$ are special cases of Eq. (A1), and can be expressed as

$$-\sum_{n=0}^{n=s-1}(-1)^n J_{2n}(Y)\Bigg),$$

$$u_{2s+1}(Y,|Y|)=(-1)^s\left(\frac{\sin(Y)}{2}-\sum_{n=0}^{n=s-1}(-1)^s J_{2n+1}(Y)\right), \tag{A5}$$

where the natural extension of the Bessel functions of negative argument has been used:

$$J_n(-Y)=(-1)^n J_n(Y). \tag{A6}$$

Equation (A5) is an extension of the results for $u_s(Y,Y)$ given by Gray and Mathews and can be derived from Eq. (26).

Substituting Eq. (A5) into Eq. (A4), and rearranging terms, we have

$$u_{2s+1}(Y,Z)=(-1)^s\left(\frac{\sin(Y)}{2}-\sum_{n=0}^{n=s-1}(-1)^n J_{2n+1}(Y)\right)\cos\left(\frac{Z^2-Y^2}{2Y}\right)+(-1)^s\left(\frac{J_0(Y)+\cos(Y)}{2}-\sum_{n=0}^{n=s}(-1)^n J_{2n}(Y)\right)\sin\left(\frac{Z^2-Y^2}{2Y}\right)-\sum_{n=0}^{n=\infty}(-1)^n\left[J_{2n+2s+1}(Y)\cos E_n\left(\frac{Z^2-Y^2}{2Y}\right)-J_{2n+2s+2}(Y)\sin E_n\left(\frac{Z^2-Y^2}{2Y}\right)\right], \tag{A7}$$

and

$$u_{2s}(Y,Z)=(-1)^s\left(\frac{J_0(Y)+\cos(Y)}{2}-\sum_{n=0}^{n=s-1}(-1)^n J_{2n}(Y)\right)\cos\left(\frac{Z^2-Y^2}{2Y}\right)-(-1)^s\left(\frac{\sin(Y)}{2}-\sum_{n=0}^{n=s-1}(-1)^n J_{2n+1}(Y)\right)\sin\left(\frac{Z^2-Y^2}{2Y}\right)-\sum_{n=0}^{n=\infty}(-1)^n\left[J_{2n+2s}(Y)\cos E_n\left(\frac{Z^2-Y^2}{2Y}\right)-J_{2n+2s+1}(Y)\sin E_n\left(\frac{Z^2-Y^2}{2Y}\right)\right], \tag{A8}$$

where

Let $s=0$ in Eq. (A7) and $s=1$ in Eq. (A8); we have

$$-2J_1(Y)\sin\left(\frac{Z^2-Y^2}{\ldots}\right)$$

$$\left(Z^2+Y^2\right)\ldots \quad n=\infty$$

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[6]B. G. Lucas and T. G. Muir, J. Acoust. Soc. Am. 72, 1289–1296
[7]W. N. Cobb, J. Acoust. Soc. Am. 75, 72–79 (1984).
[8]A. Gray and G. B. Mathews, *A Treatise on Bessel Functions And Their*